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CORRELATED COHERENT STATES - QUANTUM ANALOGUE OF THERMAL STATES

(ON THE PROBLEM OF INCORPORATING THERMODYNAMICS INTO QUANTUM THEORY)

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Abstract

In this paper, we show that correlated coherent states (CCSs) are the most adequate candidates for the role of quantum analogues of the thermal states. The main result of our study reduces to the fact that quantum thermal effects under conditions of the equilibrium of an object with the stochastic environment at any temperatures can be explained consistently only on the basis of the set of CCSs.

1. Wave functions of correlated coherent states of an arbitrary vacuum

In our paper [1], we proposed an original approach to the incorporation of stochastic thermodynamics into quantum theory. It is based on the concept of consistent inclusion of the holistic stochastic environmental action consisting of the thermal environmental action in addition to the quantum one. Bogoliubov arbitrary vacua were used as a model of the generalized thermostat. In this case, it becomes possible to bring the set of quantum states described by the complex wave function in correspondence with equilibrium thermal states. In the next paper [2], it was shown that squeezed correlated states can adequately describe only the cases of the contact with the cold vacuum, which exists at the zero Kelvin temperature, which is not significant for thermodynamics.

Now we consider a system (a quantum oscillator) under the conditions of the contact with the thermostat modeled by an arbitrary vacuum. Under these conditions, this system can be described by the complex wave function that is dependent on the parameter $\alpha \neq \frac{\pi}{2}$. [1] In the coordinate representation, in the general case, it has the form

$$\psi_\alpha(q) = [2\pi(\Delta q_0)^2 \frac{1}{\cos\alpha}]^{-1/4} \exp\left\{-\frac{q^2}{4(\Delta q_0)^2} e^{i\alpha}\right\}. \tag{1}$$

Here,

$$(\Delta q_0)^2 = \frac{\hbar}{2\gamma}, \tag{2}$$

where the coefficient $\gamma > 0$ and $\hbar$ is the Planck constant. The coordinate variance $\overline{(\Delta q_\alpha)^2} \equiv (\Delta q_\alpha)^2$ in the arbitrary-vacuum state $|\psi_\alpha\rangle$ calculated using wave function (1) has the form

$$(\Delta q_\alpha)^2 = \int_{-\infty}^{+\infty} \psi_\alpha^* q^2 \psi_\alpha dq = \frac{\hbar}{2\gamma} \cdot \frac{1}{\cos\alpha} = (\Delta q_0)^2 \cdot \frac{1}{\cos\alpha}. \tag{3}$$

[1] The states corresponding to $\alpha = 0$ were considered in our paper [2] and were qualified as squeezed correlated states (SCSs), which are inadequate to the thermal states.

Similar calculations for the momentum variance $\overline{(\Delta p_\alpha)^2} \equiv (\Delta p_\alpha)^2$ in this state $|\psi_\alpha\rangle$ lead to the result

$$(\Delta p_\alpha)^2 = \frac{\hbar\gamma}{2} \cdot \frac{1}{\cos\alpha} = (\Delta p_0)^2 \cdot \frac{1}{\cos\alpha}, \tag{4}$$

where $\dfrac{\hbar\gamma}{2} \equiv (\Delta p_0)^2$.

We show that wave function (1) is completely identical to the state of the contact with the arbitrary vacuum $\psi_{\tau,\phi}$, which was found in [3] using the Bogoliubov (u, v) - transformations from the equations

$$\hat{b}\psi_{\tau,\varphi}(q,\omega) = 0, \tag{5}$$

where the operator $\hat{b}$ is the quasiparticle annihilation operator for the arbitrary vacuum, or

$$\frac{d\psi_{\tau,\varphi}}{dq} + \frac{u-v}{u+v} \cdot \frac{\omega}{\hbar}\, q\, \psi_{\tau,\varphi} = 0 \tag{6}$$

In the general case [4], the complex functions u and v containing in (6) are determined in terms of the free parameters (τ, θ, φ) as follows:

$$u = \mathrm{ch}\,\tau \cdot e^{i\varphi}; \quad v = \mathrm{sh}\,\tau \cdot e^{-i\varphi}. \tag{7}$$

In this case, (τ, θ, φ) can be interpreted as Euler angles, which are used to parametrize the group of rotations $O\,(3)$. Without loss of generality, we assume that $\theta = 0$ in this case.

The solution of Eq. (6) for arbitrary τ and φ has the form of the complex Gaussoid

$$\psi_{\tau,\varphi}(q,\omega) = C \exp\left\{-\frac{q^2}{4(\Delta q_0)^2} \cdot \frac{u-v}{u+v}\right\}, \tag{8}$$

where $(\delta q_0)^2 \equiv \dfrac{\hbar}{2\omega}$.

If formulas (7) and the normalization conditions are taken into account, expression (8) becomes

$$\psi_{\tau,\varphi}(q,\omega) = \left[2\pi(\ \Delta q_{\tau,\varphi})^2\right]^{-1/4} \exp\left\{-\frac{q^2}{4(\Delta q_{\tau,\varphi})^2}(1 - i\beta_{\tau,\varphi})\right\}. \tag{9}$$

Here,

$$(\Delta q_{\tau,\varphi})^2 = (\Delta q_0)^2(\mathrm{ch}\,2\tau - \mathrm{sh}\,2\tau \cdot \cos 2\varphi) \tag{10}$$

$$\beta_{\tau,\varphi} = \mathrm{sh}\,2\tau \cdot \sin 2\varphi \tag{11}$$

For the convenience of comparison of different representations (1) and (9) of the wave function, we endow them with the same forms. Writing the exponent $e^{i\alpha}$ in (1) in the trigonometric form, we obtain the result

$$\psi_\alpha(q) = [2\pi(\Delta q_0)^2 \frac{1}{\cos\alpha}]^{-1/4} \exp\left\{-\frac{q^2}{4(\Delta q_0)^2}\cos\alpha(1 + i\tan\alpha)\right\}. \tag{12}$$

To make expressions (1) and (12) more similar, we assume

$$\beta_\alpha \equiv \tan\alpha \tag{13}$$

and use the value of $(\Delta q_\alpha)^2$ in accordance with (3). Finally, expression (1) can be written in the form

$$\psi_\alpha = \left[2\pi(\ \Delta q_\alpha)^2\right]^{-1/4} \exp\left\{-\frac{q^2}{4(\Delta q_\alpha)^2}(1 - i\beta_\alpha)\right\}. \tag{14}$$

It is natural to assume that formulas (1) and (14) obtained using different initial preconditions and, accordingly, expressed in terms of different parameters are nevertheless related to identical states. Then the condition for the coincidence between $(\Delta q_\alpha)^2$ (3) and $(\Delta q_{\tau,\varphi})^2$ (10) can be obtained if it is required that the following conditions be satisfied at the same time:

$$(\mathrm{ch}\, 2\tau - \mathrm{sh}\, 2\tau \cdot \cos 2\varphi) \Leftrightarrow \frac{1}{\cos\alpha} \tag{15}$$

$$\mathrm{sh}\, 2\tau \cdot \sin 2\varphi \Leftrightarrow \tan\alpha. \tag{16}$$

This turns out to be possible if in (15) $\cos 2\varphi = 0$ and in (16) $\sin 2\varphi = 1$, which agrees with the well-known assertion that the correlated coherent states (CCS) in the (u, v)-transformations are fixed by the parameter $\varphi = \frac{\pi}{4}$. Thus, we demonstrated that the states $\psi_\alpha(q)$ (1) and $\psi_{\tau,\varphi}(q)\Big|_{\varphi=\frac{\pi}{4}}$ (14) under condition $\tan\alpha = \sinh 2\tau$ agree well with each other.

2. Correlated coherent states as thermal ones

To study the possibility of endowing the states $|\psi_\alpha\rangle$ with the meaning of thermal states, it is necessary to bring the parameter α in correspondence with the temperature that has no pre-image in quantum mechanics. To do this, we consider the expression for the Planck energy

$$\mathcal{E}_{Pl} = \frac{\hbar\omega}{2} \coth\frac{\hbar\omega}{k_B T}. \tag{17}$$

We note that this formula is strictly equilibrium and corresponds to the Kelvin temperature T . Therefore, based on formula (17), we can bring the parameter $\alpha \neq \frac{\pi}{2}, 0$ in correspondence with the temperature explicitly. In what follows, we use the fact that the average values of the kinetic $\overline{K}$ and potential $\overline{U}$ energies are equal for a quantum oscillator in the thermal-equilibrium state, so that from formula (17), we obtain

$$\overline{K} = \overline{U} = \frac{\mathcal{E}_{Pl}}{2} = \frac{\hbar\omega}{4} \coth\frac{\hbar\omega}{k_B T}. \tag{18}$$

Taking into account that $\overline{K} = \frac{1}{2m}\overline{(\Delta p)^2}$ and $\overline{U} = \frac{m\omega^2}{2}\overline{(\Delta q)^2}$ and assuming that $m = 1$, from formula (18), we obtain the coordinate and momentum variances, letting the subscript T denote their relation with the Planck distribution:

$$(\Delta q_T)^2 = \frac{\hbar}{2\omega} \coth\frac{\hbar\omega}{k_B T} = (\Delta q_0)^2 \coth\frac{\hbar\omega}{k_B T}. \tag{19}$$

$$(\Delta p_T)^2 = \frac{\hbar\omega}{2} \coth\frac{\hbar\omega}{k_B T}. \tag{20}$$

As follows from formulas (19) and (20), as T increases, the coordinate and momentum variances increase synchronously for thermal states (unlike the SCSs), so that their product $(\mathcal{UP})_{pq}$ also increases in this case.

The qualitative distinction between the thermal-like SCSs ($\alpha = 0$) and the thermal states (CCS) can be demonstrated obviously by giving them the geometric interpretation in the phase plane. For convenience, we put $\hbar = 1$ and choose the dimensionless variables $\hat{\mathcal{P}} = \frac{1}{i}\frac{d}{d\mathcal{Q}}; \hat{\mathcal{Q}}$. Then, for cold-vacuum states (SCSs of the $|\psi_0\rangle$ type), the quantities $\Delta\mathcal{P}_0$ and $\Delta\mathcal{Q}_0$ (in view of their equality in accordance with formulas (3) and (4)) have the meaning of the sides of a single square with area of $1/4$. At the same time, the thermal CCSs (of the $|\psi_T\rangle$ type) in this plane correspond to different squares whose sides and areas increase consistently with temperature.

We now compare the coordinate $(\Delta q_\alpha)^2$ (3) and momentum $(\Delta p_\alpha)^2$ (4) variances calculated using wave function (1) with their corresponding values $(\Delta q_T)^2$ (19) and $(\Delta p_T)^2$ (20) obtained using the Planck energy. They turn out to be completely identical if we set

$$\gamma = \omega \tag{21}$$

and

$$\frac{1}{\cos\alpha} = \coth\frac{\hbar\omega}{k_B T}. \tag{22}$$

Thus, in accordance with relation (22), the parameter α in formula (1) fixes the states corresponding to the equilibrium at the temperature T.

Expression (22) allows representing the exponent $e^{i\alpha}$ in formula (1) in the trigonometric form, explicitly indicating its relation with the temperature T in this case. Calculating $\sin\alpha = \sqrt{1 - \tanh^2\frac{\hbar\omega}{k_B T}} = \dfrac{1}{\cosh\frac{\hbar\omega}{k_B T}}$ in advance, we obtain

$$e^{i\alpha} = \cos\alpha + i\sin\alpha = \tanh(\frac{\hbar\omega}{k_B T}) + i\frac{1}{\cosh(\frac{\hbar\omega}{k_B T})} = \tanh(\frac{\hbar\omega}{k_B T})\left[1 + i\frac{1}{\sinh\frac{\hbar\omega}{k_B T}}\right] \tag{23}$$

Returning to wave function (1), we can demonstrate the explicit temperature dependence of its amplitude and phase by labeling it with the subscript T

$$\psi_T(q) = [2\pi(\Delta q_0)^2\coth\frac{\hbar\omega}{k_B T}]^{-1/4}\exp\left\{-\frac{q^2}{4(\Delta q_0)^2}\tanh(\frac{\hbar\omega}{k_B T})[1 + i\frac{1}{\sinh\frac{\hbar\omega}{k_B T}}]\right\}. \tag{24}$$

3. Saturation of the Schrödinger UR in the correlated coherent states

It is interesting to analyze certain CCS peculiarities. As is well known, the most general *Schrödinger uncertainties relation* (SUR) for the coordinate and momentum has the form

$$\Delta p\cdot\Delta q \geqslant \left|\langle\psi|\delta\hat{p}\cdot\delta\hat{q}|\psi\rangle\right|. \tag{25}$$

Here, the left-hand side of the inequality contains the product of the momentum and coordinate uncertainties in the state $|\psi\rangle$ calculated using the definition

$$(\Delta p)^2 \equiv \langle\psi|(\delta\hat{p})^2|\psi\rangle; \quad (\Delta q)^2 \equiv \langle\psi|(\delta\hat{q})^2|\psi\rangle. \tag{26}$$

The expression in the right-hand side

$$\left|\langle\psi|\delta\hat{p}\cdot\delta\hat{q}|\psi\rangle\right| = \left|\langle\delta p|\delta q\rangle\right| \tag{27}$$

has the meaning of the correlator of object momentum and coordinate fluctuations in the same state, which is expressed in terms of the fluctuation operators $\delta\hat{p}$ and $\delta\hat{q}$. We recall that the left- and right-hand sides of relation (22) must be calculated independently.

In the arbitrary-vacuum state, the equality for the means

$$(\Delta p_\alpha)^2 = \gamma^2 (\Delta q_\alpha)^2 \tag{28}$$

is valid [5] so that the *left-hand* side of SUR (25) becomes equal to

$$\Delta p_\alpha \cdot \Delta q_\alpha = \gamma \Delta q_\alpha^2. \tag{29}$$

Calculating the correlator in the *right-hand* side of SUR (25) with respect to the state $|\psi_\alpha\rangle$, we obtain

$$\Big|\langle\psi_\alpha|\hat{p}\cdot\hat{q}|\psi_\alpha\rangle\Big| = \sqrt{\frac{\hbar^2}{4}\tan^2\alpha + \frac{\hbar^2}{4}} = \frac{\hbar}{2}\cdot\frac{1}{\cos\alpha}. \tag{30}$$

We note that the correlator here is expressed in terms of the α - phase of the wave function. Taking into account that in accordance with (3), $\dfrac{1}{\cos\alpha} = \dfrac{2}{\hbar}\cdot\gamma\Delta q_\alpha^2$, we reduce correlator (30) to the final form

$$\Big|\langle\psi_\alpha|\hat{p}\cdot\hat{q}|\psi_\alpha\rangle\Big| = \gamma\Delta q_\alpha^2, \tag{31}$$

which completely coincides with product (29) of uncertainties, i.e., with the left-hand side of the SUR (25).

Thus, we see that the correlated state $|\psi_\alpha\rangle$ is outlined by the fact that the SUR in it indeed acquires the form of the *equality*, i.e., becomes saturated

$$\Delta p_\alpha \cdot \Delta q_\alpha = \Big|\langle\psi_\alpha|\hat{p}\cdot\hat{q}|\psi_\alpha\rangle\Big|. \tag{32}$$

The interpretation of the radicand in (30) can be related to that of a similar expression for the cold-vacuum state ψ_0 . For $\alpha = 0$, saturated SUR (30) transforms into the saturated Heisenberg UR

$$\Delta p_0 \cdot \Delta q_0 = \big|\langle\psi_0|\frac{1}{2}[\hat{p},\hat{q}]|\psi_0\rangle\big| = \frac{\hbar}{2}. \tag{33}$$

Here, as is known, $\dfrac{\hbar}{2}$ is the measure of a purely quantum environmental action occurring in the cold vacuum. Thus, for $\alpha = 0$, the correlator $\big|\langle\psi_\alpha|\hat{p}\cdot\hat{q}|\psi_\alpha\rangle\big|$ has a value that is as minimum as possible. As was expected, the state $|\psi_0\rangle$ indeed has the meaning of the state of equilibrium with the cold vacuum because it corresponds to the minimum value of the vacuum energy $\dfrac{\hbar\omega}{2}$.

It is adopted to regard the Heisenberg relation as a fundamental equality reflecting the presence of unavoidable purely quantum effects in the Nature. We assume that in comparison with (33), the origin of the additional term in the radicand of correlator (30) in the form $\dfrac{\hbar^2}{4}\tan^2\alpha$ is related to precisely the inclusion of the thermal influence of the arbitrary vacuum, which is manifested in the complex character of the wave function. The fact that the macroparameter of the effective quantum-thermostat action

$$\mathbb{J}_\alpha \equiv \sqrt{\frac{\hbar^2}{4}\tan^2\alpha + \mathbb{J}_0^2} = \frac{\hbar}{2}\frac{1}{\cos\alpha}, \tag{34}$$

(here, $\mathbb{J}_0^2 = \dfrac{\hbar}{2}$ relates to the purely quantum influence) which was introduced previously in the framework of stochastic thermodynamics [6], coincides with the right-hand side of formula (30) is evidence in favor of this assertion. This fact gives grounds to assume from now on that for the arbitrary vacuum, saturated SUR (32) also corresponds to the more general equilibrium state $|\psi_\alpha\rangle$ occurring at the simultaneous presence of the quantum and thermal actions. Thus, among the functions providing the saturation of SUR (32), there exist the functions $\psi_\alpha(q)\Big|_{\alpha\neq 0,\frac{\pi}{2}}$, which allow taking into account extra thermal effects in addition to the quantum ones in a certain temperature range. Thus, the CCSs can be regarded as quantum analogues of thermal states in a sufficiently substantiated way.

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